LII. *A Demonſtration of the Second Rule in the Eſſay towards the Solution of a Problem in the Doctrine of Chances, publiſhed in the Philoſophical Tranſactions*, Vol. LIII. *Communicated by the Rev. Mr.* Richard Price, *in a Letter to Mr.* John Canton, *M. A. F. R. S.*

Dear Sir, Nov. 26, 1764.

Read Dec. 6, 1764.

I Send you the following *Supplement to the Eſſay on a Problem in the Doctrine of Chances*, hoping that you may not think it improper to be communicated to the Royal Society. I ſhould not have troubled you again in this way had I not found that ſome additions to my former papers were neceſſary in order to explain ſome paſſages in them, and particularly what is hinted in the note at the end of the Appendix. " I have firſt given the deduction of Mr. Bayes's ſe- " cond rule chiefly in his own words; and then " added, as briefly as poſſible, the demonſtrations of " ſeveral propoſitions, which ſeem to improve con- " ſiderably the ſolution of the problem, and to throw " light on the nature of the curve by the quadrature " of which this ſolution is obtained." Perhaps, there is no reaſon for being very anxious about proceeding to further improvements. It would, however, be very agreeable to me to ſee a yet eaſier and nearer approximation to the value of the two ſeries's in the firſt rule: but this I muſt leave abler perſons to ſeek, chuſing now entirely to drop this ſubject.

The

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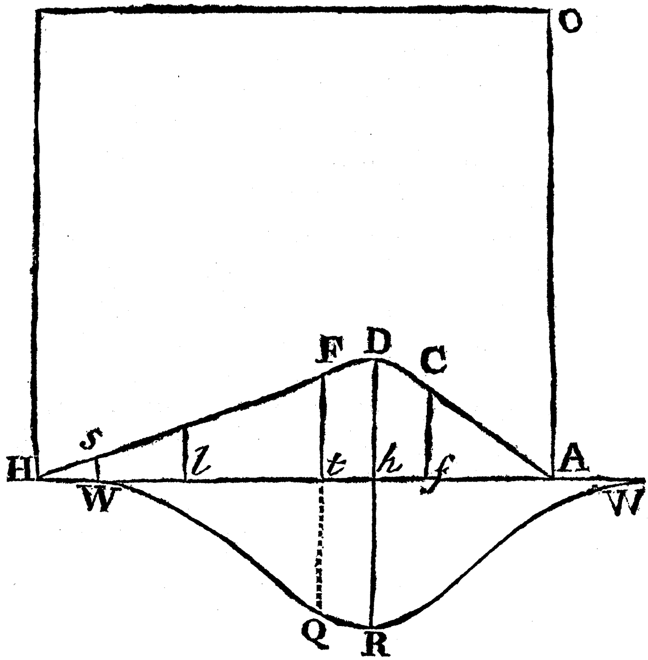
The ſolution of the problem enquired after in the papers I have ſent you has, I think, been hitherto a *deſideratum* in philoſophy of ſome conſequence. To this we are now in a great meaſure helped by the abilities and ſkill of our late worthy friend; and thus are furniſhed with a neceſſary guide in determining the nature and proportions of unknown cauſes from their effects, and an effectual guard againſt one great danger to which philoſophers are ſubject; I mean, the danger of founding concluſions on an inſufficient induction, and of receiving juſt concluſions with more aſſurance than the number of experiments will warrant. I am, under a ſenſe of the value of your friendſhip, heartily yours,

Richard Price.



ART. 1. If the curve ADH be divided into two parts by the ordinate Dh making Ah to Hh as p is to q; then taking $a = \frac{p}{n}$ and $b = \frac{q}{n}$ the ratio of the Area ADh to HO will be $\frac{a^{p+1} \times b^q}{p+1} \times$

$$\overline{1 + \frac{q}{p+2} \times \frac{p}{q} + \frac{q \times \overline{q-1} \times p^2}{\overline{p+2} \times \overline{p+3} \times q^2} + \frac{q \times \overline{q-1} \times \overline{q-2} \times p^3}{\overline{p+2} \times \overline{p+3} \times \overline{p+4} \times q^3}}$$

+ &c. For the series $\frac{x^{p+1}\, r^q}{p+1} + \frac{q}{p+1} \times \frac{x^{p+2} \times r^{q-1}}{p+2}$ + &c. in Prop: 10. Art. 2. of the Essay, which expresses the ratio of ACf to HO, becomes this series when $x = a = \frac{p}{n}$, $b = r = \frac{q}{n}$; that is when

Cf

C f has moved till it coincides with D b and A C f becomes A D b. In like manner, from Art. 3. in the Essay, it appears that the ratio of H D b to H O is $\frac{a^p b^{q+1}}{q+1} \times \overline{1 + \frac{p}{q+2} \times \frac{q}{p} + \frac{p}{q+2} \times \frac{p-1}{q+3} \times \frac{q^2}{p^2}}$ + &c.

From hence it follows that the ratio of the difference between A D b and H D b to H O is $\frac{a^p b^q}{n}$ multiplied by the difference between the series $\frac{p}{p+1} + \frac{q}{p+1} \times \frac{p^2}{pq+2q} + \frac{q \times \overline{q-1} \times p^3}{\overline{p+1} \times \overline{p+2} \times \overline{pq^2+3q^2}}$ + &c. and the series $\frac{q}{q+1} + \frac{p \times q^2}{\overline{q+1} \times \overline{pq+2p}} + \frac{p \times \overline{p-1} \times q^3}{\overline{q+1} \times \overline{q+2} \times \overline{p^2 q + 3p^2}}$ + &c. the former series being to be substracted from the latter, if H D b is greater than A D b, and *vice versa*.

2. The ratio of any term in the former of the two foregoing series to that which next but one follows the correspondent term in the latter is $\frac{pq+p}{p \times q} \times \frac{pq+2p}{p \times q} \times \frac{p \times q}{qp+q} \times \frac{pq+3p}{pq-q} \times \frac{pq}{pq+2q} \times \frac{pq+4p}{pq-2q} \times \frac{pq-p}{pq+3q} \times \frac{pq+5p}{pq-3q} \times \frac{pq-2p}{pq+4q} \times \frac{pq+6p}{pq-4q}$ &c. taking twice as many terms and four over as there are units in the number which expresses the distance of the term in the former series from its first term; which

 ratio

ratio if q be greater than p, it is evident muſt be greater than the ratio of equality. Wherefore, if from the ſecond ſeries you ſubſtract the two firſt terms which together are leſs than two, the remainder is leſs than the former ſeries; and of conſequence, the former ſeries ſubſtracted from the latter cannot leave a remainder ſo great as two. And therefore in this caſe, that is, when q is greater than p, by the preceding article, the ratio of H D b—A D b to H O cannot be ſo great as $\frac{2a^p b^q}{n}$.

3. The curve A D H being as before divided into two parts A D b and H D b, let the ordinates C f and F t be placed on each ſide of D b and at the ſame diſtance from it, and let z be the ratio of $b f$ or $b t$ to A H. Then if y, x and r be reſpectively the ratios of C f, A f and H f to A H, by the nature of the curve $y = x^p r^q$. But becauſe the ratio of A b to A H is a, and that of $b f$ to A H is z, the ratio of A b—$b f$ (=A f) to A H is $a-z$. Wherefore $a-z=x$. And in like manner $b+z=r$. But $y=x^p r^q$, and y is the ratio of C f to A H. Wherefore the ratio of C f to A H is $\overline{a-z}|^p \times \overline{b+z}|^q$. And in like manner the ratio of F t to A H is $\overline{a+z}|^p \times \overline{b-z}|^q$. And conſequently C f is to F t as $\overline{a-z}|^p \times \overline{b+z}|^q$ is to $\overline{a+z}|^p \times \overline{b-z}|^q$.

4. If q is greater than p, $\overline{a+z}|^p \times \overline{b-z}|^q$ is greater than $\overline{a-z}|^p \times \overline{b+z}|^q$, and the ratio between them increaſes as z increaſes. For the hyperbolic logarithm of

of that ratio taken as usual, and then instead of p and q putting $n\,a$ and $n\,b$ because $a = \frac{p}{n}$ and $b = \frac{q}{n}$ (Vid. Art. 1.) you will find to be $2\,n$ multiplied by the series $\frac{b^2 - a^2}{3\,b^2\,a^2} \times z^3 + \frac{b^4 - a^4}{5\,b^4\,a^4} \times z^5 + \frac{b^6 - a^6}{7\,b^6\,a^6} \times z^7$ &c. which logarithm when q is greater than p, and therefore b greater than a has all its terms positive, and so much the greater as z is greater; and therefore it is the logarithm of a ratio greater than that of equality, and which increases as z increases.

5. By Art. 3. F t is to C f as $\overline{a+z}|^{p} \times \overline{b-z}|^{q}$ is to $\overline{a-z}|^{p} \times \overline{b+z}|^{q}$. And by Art. 4. $\overline{a+z}|^{p} \times \overline{b-z}|^{q}$ is greater than $\overline{a-z}|^{p} \times \overline{b+z}|^{q}$, and the ratio between them increases as z increases, if q is greater than p. Wherefore, upon this supposition, also F t is greater than C f, and the ratio between them increases as z or $h\,t$ and $h\,f$ increases, and consequently this will be true also concerning the areas described by them as their equal abscisses $h\,t$ and $h\,f$ increase. Wherefore, when q is greater than p, D $h\,t$ F is greater than D $h\,f$ C, and the ratio between them increases as $h\,f = h\,t$ increases.

6. Because A h is to H h as p is to q, when q is greater than p, H h is greater than A h. In H h therefore taking $h\,l$ equal to A h, by the preceding Art. the part of the figure H D h which insists upon $h\,l$ will be greater than A D h, and the ratio of that part of H D h to A D h will be greater than the ratio of D $h\,t$ F to D $h\,f$ C. Consequently, much more (q being greater than p) the whole figure H D h is greater

greater than A D *b*, and the ratio of H D *b* to A D *b* is greater than that of D *b t* F to D *b f* C.

7. When q is greater than p, $\overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$ is greater than $\overline{1 - \frac{nz}{p}}|^{p} \times \overline{1 + \frac{nz}{q}}|^{q}$ and less than $\overline{1 - \frac{nz}{q}}|^{q} \times \overline{1 + \frac{nz}{p}}|^{p}$. For the fluxion of $\overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$ is $-\frac{n^3 z \dot{z}}{pq} \times \overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}-1}$ and the fluxion of $\overline{1 - \frac{nz}{p}}|^{p} \times \overline{1 + \frac{nz}{q}}|^{q}$ (because $p + q = n$) is $-\frac{n^3 z \dot{z}}{pq} \times \overline{1 - \frac{nz}{p}}|^{p-1} \times \overline{1 + \frac{nz}{q}}|^{q-1}$. Wherefore $\overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$ is to $\overline{1 - \frac{nz}{p}}|^{p} \times \overline{1 + \frac{nz}{q}}|^{q}$ as the fluxion of the former multiplied by $\overline{1 - \frac{n^2 z^2}{pq}}|$ to the fluxion of the latter multiplied by $(\overline{1 - \frac{nz}{p}} \times \overline{1 + \frac{nz}{q}}$ or) $1 - \frac{nz}{p} + \frac{nz}{q} - \frac{n^2 z^2}{pq}$. From which analogy, because q is greater than p, it is plain that $\overline{1 - \frac{nz}{p}}|^{p} \times \overline{1 + \frac{nz}{q}}|^{q}$ varies at a greater rate in respect of its own magnitude than $\overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$ does. And, because their fluxions as found out before have a negative sign before them, they both decrease as z increases;

creaſes; conſequently, if they are ever equal, as z increaſes the latter muſt be the largeſt. But when $z = 0$ they are each equal to 1. In like manner the other part of this article appears. And hence, ſince $a = \frac{p}{n}$ and $b = \frac{q}{n}$, it is manifeſt that $a^p b^q \times \overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$ is greater than $\overline{a - z}|^p \times \overline{b + z}|^q$ and leſs than $\overline{a + z}|^p \times \overline{b - z}|^q$, when q is greater than p.

8. Suppoſe now further that the curve R Q W be deſcribed meeting the lines D b, F t, $b\,t$ produced in R, Q, W, in ſuch manner that F t, which is to C f as $\overline{a + z}|^p \times \overline{b - z}|^q$ to $\overline{a - z}|^p \times \overline{b + z}|^q$ (Vid. Art. 3.) ſhall be to Q t as $\overline{a + z}|^p \times \overline{b - z}|^q$ to $a^p b^q \times \overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$ wherever the points t and f fall at equal diſtances from b. And it is manifeſt by the foregoing Art. that Q t muſt always be greater than C f, and leſs than F t. And of conſequence the ſame muſt be true concerning the areas deſcribed by their motion while their equal abſciſſes increaſe. Wherefore R b t Q is greater than D b f C, and leſs than D b t F.

9. Since F t is to Q t as $\overline{a + z}|^p \times \overline{b - z}|^q$ to $a^p b^q \times \overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$; and $\overline{a + z}|^p \times \overline{b - z}|^q$ (by

Art.

Art. 3.) expresses the ratio of Ft to A H; the ratio of Qt to A H must be $a^p b^q \times \overline{1 - \frac{n^2 z^2}{pq}}|^{\frac{n}{2}}$, and as has been all along supposed z is the ratio of bt to A H. Wherefore, by squaring the curve R $b\,t$ Q, it will appear that the ratio of R $b\,t$ Q to H O is $a^p b^q \times \overline{z - \frac{n^3 z^3}{2.3pq} + \frac{n-2}{4} \times \frac{n^5 z^5}{2.5p^2q^2} - \frac{n-2}{4} \times}$ $\overline{\frac{n-4}{6} \times \frac{n^7 z^7}{2.7p^3q^3} + \&c.}$ which $\left(\text{if } m^2 = \frac{n^3}{2pq}\right)$ is $a^p b^q \times \frac{\sqrt{2pq}}{n\sqrt{n}} \times \overline{mz - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5} - \frac{n-2}{2n}}$ $\overline{\times \frac{n-4}{3n} \times \frac{m^7 z^7}{7} + \frac{n-2}{2n} \times \frac{n-4}{3n} \times \frac{n-6}{4n} \times \frac{m^9 z^9}{9}}$ $- \&c.$ Which last series when $\frac{n^2 z^2}{pq} = 1$, and consequently the ordinate Qt vanishes, becomes $B - \frac{B^3}{3} + \frac{B^2-1}{2B^2} \times \frac{B^5}{5} - \frac{B^2-1}{2B^2} \times \frac{B^2-2}{3B^2} \times \frac{B^7}{7} + \&c.$ taking $B^2 = \frac{n}{2}$.

10. If $B^2 = \frac{n}{2}$ the ratio of the whole figure R Q W b to H O is $\frac{\sqrt{2pq}}{n\sqrt{n}} \times a^p b^q \times \overline{B - \frac{B^3}{3} + \frac{B^2-1}{2B^2} \times \frac{B^5}{5} - \&c.}$ Now, (by Prop. 10. Art. 4. of the Essay) the ratio of A C F H to H O is $\frac{1}{n+1}$

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$\times \frac{1}{E}$, E being the coefficient of that term of the binomial $\overline{a+b}^n$ expanded in which occurs $a^p b^q$: Wherefore, the ratio of R Q W b to A C F H is $\frac{n+1}{n} \times \frac{\sqrt{2pq}}{\sqrt{n}} \times E a^p b^q \times \overline{B - \frac{B^3}{3} + \frac{B^2 - 1}{2B^2} \times \frac{B^5}{5} \&c.}$ Put G now for the coefficient of the middle term of the same binomial, and if $p = q = \frac{n}{2}$, $E = G$, $a = \frac{1}{2} = b$ the area R Q W b is equal to half A C F H; for then Q t, F t, C f are all equal, and consequently the areas R Q W b, H D b and A D b. Wherefore, the series $B - \frac{B^3}{3} + \&c.$ is equal to $\frac{\sqrt{2n}}{n+1} \times \frac{2^{n-1}}{G}$. But the series $B - \frac{B^3}{3} + \&c.$ (because $B^2 = \frac{n}{2}$) does not alter whatever p and q are, whilst their sum n remains the same. Wherefore, in all cases, the ratio of R Q W b to A C F H is $\frac{\sqrt{pq}}{n} \times \frac{E a^p b^q}{G} \times 2^n$.

11. By Prop. 10. Art. 4. of the Essay, the ratio of A C F H to H O* is $\frac{1}{n+1} \times \frac{1}{E}$; and by Art. 9. the ratio of R b t Q to H O is $a^p b^q \times \frac{\sqrt{2pq}}{n\sqrt{n}} \times \overline{m z - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5} \&c.}$ Wherefore, the ratio of

* It is hoped that the imperfection of the figure all along referred to will be excused. The lines R b and D b should appear equal; and it will be found presently, that the curve line A C D F H should have been drawn from F and C convex towards A H.

RbtQ to ACFH is $\frac{n+1}{n} \times \frac{\sqrt{2pq}}{\sqrt{n}} \times E a^p b^q \times \overline{mz - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5} - \frac{n-2}{2n} \times \frac{n-4}{3n} \times \frac{m^7 z^7}{7} + \&c.}$ Likewise, by Art. 10. the ratio of RQWb to ACFH is $\frac{\sqrt{pq}}{n} \times \frac{E a^p b^q}{G} \times 2^n$. Wherefore the ratio of RbtQ to RQWb is $\frac{n+1}{\sqrt{n}} \times \frac{\sqrt{2}}{2^n} \times G \times \overline{mz - \frac{m^3 z^3}{3} + \&c.}$

12. By Art. 2. 6. When q is greater than p, the ratio of HDb — ADb to HO is less than $\frac{2a^p b^q}{n}$. And by Prop. 10. Art. 4. of the Essay, the ratio of ACFH or HDb + ADb to HO is $\frac{1}{n+1} \times \frac{1}{E}$. Wherefore, the sum of these two ratios, or the ratio of 2 HDb to HO, is less than $\frac{1}{n+1} \times \frac{1}{E} + \frac{2a^p b^q}{n}$; and the difference between them, or the ratio of 2 ADb to HO is greater than $\frac{1}{n+1} \times \frac{1}{E} - \frac{2a^p b^q}{n}$. Wherefore, the ratio of 2 HDb to 2 ADb, or that of HDb to ADb, is less than that of $\frac{1}{n+1} \times \frac{1}{E} + \frac{2ab^{pq}}{n}$ to $\frac{1}{n+1} \times \frac{1}{E} - \frac{2a^p b^q}{n}$, which is equal to the ratio of $1 \times 2 E a^p b^q + \frac{2 E a^p b^q}{n}$ to 1

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$-2Ea^pb^q - \frac{2Ea^pb^q}{n}$. But the ratio of H D h to A D h, by Art. 6. is greater than the ratio of D h t F to D h f C, when q is greater than p. Wherefore, much more when q is greater than p, the ratio of D h t F to D h f C will be lefs than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. And becaufe, by Art. 8. R h t Q is a mean between D h t F and D h f C, the ratio of D h t F to R h t Q will be lefs than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. And the ratio of D h f C to R h t Q will be greater than that of $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$ to $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$.

RULE II.

If nothing is known of an event but that it has happened p times and failed q in $p+q$ or n trials, and q be greater than p; and from hence I judge that the probability of its happening in a fingle trial lies between $\frac{p}{n}$ and $\frac{p}{n}+z$, (if $m^2 = \frac{n^3}{2pq}$, $a = \frac{p}{n}$, $b = \frac{q}{n}$, E the coefficient of the term in which occurs a^pb^q when $\overline{a+b}|^n$ is expanded, and $\Sigma =$ $\frac{n+1}{n} \times \frac{\sqrt{2pq}}{\sqrt{n}} \times Ea^pb^q \times \overline{mz - \frac{m^3z^3}{3} + \frac{n-2}{2n} \times \frac{m^5z^5}{5} -}$

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$\frac{n-2}{2n} \times \frac{n-4}{3n} \times \frac{m^7 z^7}{7} +$ &c.) my chance to be in the right is greater than Σ, and less than $\Sigma \times \frac{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}{1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}}$. For by Art. 11. compared with the value of Σ here set down, the ratio of R*bt*Q to ACFH is Σ. But by Art. 8. D*bt*F is greater than R*bt*Q, and by Art. 12. the ratio of D*bt*F to R*bt*Q is less than that of $1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}$ to $1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}$. From whence it is plain that the ratio of D*bt*F to ACFH is greater than Σ, and less than $\Sigma \times \frac{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}{1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}}$. But, as appears from the 10th Proposition in the Essay, the chance that the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} + z$ (that is, between the ratio of A*b* to AH, and that of A*t* to AH) is the ratio of D*bt*F to ACFH. Wherefore, the chance I am right in my guess is greater than Σ and less than $\Sigma \times$

$$\frac{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}{1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}}.$$

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In like manner, 2dly, the ſame things ſuppoſed, if I judge that the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} - z$, my chance to be right is leſs than Σ, and greater than $\Sigma \times \frac{1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}}{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}$. This is manifeſt as the other caſe, becauſe DbfC is leſs than RbtQ, but the ratio between them is greater than that of $1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}$ to $1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}$.

3dly, If, other things ſuppoſed as before, p is greater than q, and I judge the probability of the event lies between $\frac{p}{n}$ and $\frac{p}{n} + z$, my chance to be right is leſs than Σ, and greater than $\Sigma \times \frac{1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}}{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}$. But if I judge it lies between $\frac{p}{n}$ and $\frac{p}{n} - z$, my chance is greater than Σ, and leſs than $\Sigma \times \frac{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}{1 - 2Ea^p b^q - \frac{2Ea^p b^q}{n}}$. And if $p = q$,

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which ever of these ways I guess, my chance is Σ exactly. This may be proved in the same manner with the foregoing cases, where q is greater than p, or may be proved from them by considering the happening and failing of an event, as the same with the failing and happening of its contrary.

4thly, Other things supposed the same, whether q be greater or less than p, and I judge that the probability of the event lies between $\frac{p}{n} + z$ and $\frac{p}{n} - z$, my chance is greater than $\frac{2\,\Sigma}{1 + 2\,E\,a^p\,b^q + \frac{2\,E\,a^p\,b^q}{n}}$, and less than $\frac{2\,\Sigma}{1 - 2\,E\,a^p\,b^q - \frac{2\,E\,a^p\,b^q}{n}}$. This is an evident corollary from the cases already determined. And here, if $p = q$, my chance is 2 Σ exactly.

Thus far I have transcribed Mr. Bayes.

It appears, from the Appendix to the Essay, that the rule here demonstrated, though of great use, does not give the required chance within limits sufficiently narrow. It is therefore necessary to look out for a contraction of these limits; and this, I think, we shall discover by the help of the following deductions; which, for the sake of greater dinstinctness, I shall give as a continuation of the foregoing Articles.

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13. The ratio of the fluxion of $\overline{1-\frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}}$ to the fluxion of $\overline{1+\frac{nz}{p}}\Big|^{p} \times \overline{1-\frac{nz}{q}}\Big|^{q}$ is $\dfrac{\overline{1-\frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}-1}}{\overline{1+\frac{nz}{p}}\Big|^{p-1} \times \overline{1-\frac{nz}{q}}\Big|^{q-1}}$; and the ratio of the fluxion of $\overline{1-\frac{nz}{p}}\Big|^{p} \times \overline{1+\frac{nz}{q}}\Big|^{q}$ to the fluxion of $\overline{1-\frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}}$ is $\dfrac{\overline{1-\frac{nz}{p}}\Big|^{p-1} \times \overline{1+\frac{nz}{q}}\Big|^{q-1}}{\overline{1-\frac{n^2 z^2}{pq}}\Big|^{\frac{n}{2}-1}}$.

This will immediately appear from Art. 7.

14. While z is increasing from nothing till $\frac{n^2 z^2}{pq}$ becomes equal to unity, these two ratios are at first greater than the ratio of equality, and increase as z increases, till they come to a *maximum*. Afterwards they decrease untill they become first equal to the ratio of equality, and then less. This is proved by finding the hyperbolic logarithms of these ratios. The hyperbolic logarithm of the first is the series $\overline{\frac{q-1}{q}-\frac{p-1}{p}} \times nz + \overline{\frac{q-1}{q^2}+\frac{p-1}{p^2}-\frac{n-2}{pq}} \times \frac{n^2 z^2}{2} + \overline{\frac{q-1}{q^3}-\frac{p-1}{p^3}} \times \frac{n^3 z^3}{3} + \overline{\frac{q-1}{q^4}+\frac{p-1}{p^4}-\frac{n-2}{p^2 q^2}} \times \frac{n^4 z^4}{4} + \overline{\frac{q-1}{q^5} -}$

4.

$-\overline{\frac{p-1}{p^5}} \times \frac{n^5 z^5}{5} + \overline{\frac{q-1}{q^6} + \frac{p-1}{p^6} - \frac{n-2}{p^3 q^3}} \times \frac{n^6 z^6}{6}$ + *&c.* The hyperbolic logarithm of the ſecond ratio is the ſeries $\overline{\frac{q-1}{q} - \frac{p-1}{p}} \times nz - \overline{\frac{q-1}{q^2} + \frac{p-1}{p^2} - \frac{n-2}{pq}} \times \frac{n^2 z^2}{2} + \overline{\frac{q-1}{q^3} - \frac{p-1}{p^3}} \times \frac{n^3 z^3}{3} - \overline{\frac{q-1}{q^4} + \frac{p-1}{p^4} - \frac{n-2}{p^2 q^2}} \times \frac{n^4 z^4}{4}$ + *&c.* It will appear from examining theſe two ſerieſes (q all along ſuppoſed greater than p) that while z is ſmall the value of each of them is poſitive, and increaſes as z increaſes till it becomes a *maximum*, after which it decreaſes till it becomes nothing, and after that negative; which demonſtrates this article.

15. The former of the two ratios in Art. 13. (q being greater than p) is at firſt, while z is increaſing from nothing, leſs than the ſecond ratio; and does not become equal to it, till ſome time after both ratios have been the greateſt poſſible.

Upon conſidering the two ſerieſes in the laſt Art. it will appear that the firſt term of the firſt ſeries is always poſitive, the ſecond negative, the third alſo negative, after which the terms become alternately poſitive and negative. On the other hand, it will appear that in the ſecond ſeries the two firſt terms are always poſitive, and all that follow negative. But as the ſerieſes converge very faſt when z is ſmall, the ſecond term being negative in the firſt ſeries and poſitive in the ſecond, has a greater effect in making the firſt ſeries leſs than the ſecond, than can be compenſated for by the terms being afterwards alternately negative and

 poſitive

positive in the one, and all negative in the other. It will further appear from considering these serieses, that the first must continue less than the second 'till z becomes so large as to make the fourth term equal to the second, in which circumstances the two serieses are nearly equal. Afterwards, as z goes on to increase, the value of both lessens continually; but the second now decreasing fastest, as before it increased fastest, becomes first nothing. After which, the other series becomes nothing; and after that both remain negative. From hence it is easy to infer this Article.

16. What has been now shewn of the ratio of the fluxion of $\overline{1-\frac{n^2z^2}{pq}}|^{\frac{n}{2}}$ to the fluxion of $\overline{1+\frac{nz}{p}}|^{p} \times \overline{1-\frac{nz}{q}}|^{q}$ compared with the ratio of the fluxion of $\overline{1-\frac{nz}{p}}|^{p} \times \overline{1+\frac{nz}{q}}|^{q}$ to the fluxion of $\overline{1-\frac{n^2z^2}{pq}}|^{\frac{n}{2}}$ is also true of the ratio of the fluxion of $a^p b^q \times \overline{1-\frac{n^2z^2}{pq}}|^{\frac{n}{2}}$ (or Qt in the figure) to the fluxion of $\overline{a+z}|^{p} \times \overline{b-z}|^{q}$ (or F t) compared with the ratio of the fluxion of $\overline{a-z}|^{p} \times \overline{b+z}|^{q}$ (or Cf) to the fluxion of $a^p b^q \times \overline{1-\frac{n^2z^2}{pq}}|^{\frac{n}{2}}$ or Qt; the latter quantities being only the former multiplied by the common and permanent quantity $a^p b^q$. It appears, therefore, that if we conceive F t, Qt, Cf (Vid. Fig.)

Fig.) to move with equal and uniform velocities, from D b and R b along A H, in order to generate the areas H D b, R W b, A D b; C f will at firſt not only decreaſe faſter than Q t, and Q t than F t; but the ratio of the rate at which C f decreaſes to the rate at which Q t decreaſes, will be greater than the ratio of the rate at which Q t decreaſes to the rate at which F t decreaſes. It appears alſo that after ſome time, firſt C f and Q t, and then Q t and F t will come to decreaſe equally; after which, Q t will decreaſe faſter than C f, and F t faſter than Q t.

17. The curves D F H, R Q W, D C A, have each of them a point of contrary flexure; and the value of z, or of the equal abſciſſes at that point, is in all three $\frac{\sqrt{pq}}{\sqrt{n^3 - n^2}}$. This may be found in the common manner, by putting the ſecond fluxions of the ordinates equal to nothing. In the ſingle caſe, when either p or q is equal to unity, one of theſe points vaniſhes, or coincides with A or H.

18. At the points of contrary flexure (q being greater than p) the ratio of the fluxion of Q t to the fluxion of F t is a *maximum*; and the ſame is true of the ratio of the fluxion of C f to the fluxion of Q t. This is found by making the fluxions of the logarithms of theſe ratios, or of

$$\frac{\overline{1 - \frac{n^2 z^2}{pq}}^{\frac{n}{2} - 1}}{\overline{1 + \frac{nz}{p}}^{p-1} \times \overline{1 - \frac{nz}{q}}^{q-1}}, \text{ and } \frac{\overline{1 - \frac{nz}{p}}^{p-1} \times \overline{1 + \frac{nz}{q}}^{q-1}}{\overline{1 - \frac{n^2 n^2}{pq}}^{\frac{n}{2} - 1}}$$

I equal

equal to nothing: which will give the value of z equal to $\frac{\sqrt{pq}}{\sqrt{n^3 - n^2}}$, or the ſame with the value of z at the points of contrary flexure.

19. At the points of contrary flexure, the ratio of the fluxion of C f to the fluxion of Q t, is greateſt in compariſon of the ratio of the fluxion of Q t to the fluxion of F t. This is proved by finding the value of z when the fluxion of the former ratio divided by the latter, or of $\frac{\overline{1 - \frac{n^2 z^2}{p^2}}\Big|^{p-1} \times \overline{1 - \frac{n^2 z^2}{q^2}}\Big|^{q-1}}{\overline{1 - \frac{n^2 z^2}{pq}}\Big|^{n-2}}$ is nothing, which will ſtill give $z = \frac{\sqrt{pq}}{\sqrt{n^3 - n^2}}$. The reaſon, therefore, in the nature of the curve, which, as the ordinates flow, keeps at firſt the exceſs of F t above Q t leſs than the exceſs of Q t above C f, operates with its greateſt force at the points of contrary flexure.

20. The greateſt part of the area R Q W h lies between R h, and the ordinate at the point of contrary flexure. By Art. 11 the ratio of R h t Q to R Q W h is $\frac{n+1}{\sqrt{n}} \times \frac{\sqrt{2}}{2^n} \times G \times \overline{mz - \frac{m^3 z^3}{3} + \frac{n-2}{2n} \times \frac{m^5 z^5}{5} - \&c.}$ Subſtitute here $\sqrt{\frac{pq}{n^3 - n^2}}$ for z, and $\frac{2^n}{\sqrt{nK} \times H}$ * for G (K being the ratio of the quadrantal arc to

* This is always the true value of G; but it would be too tedious to give the demonſtration of this here.

radius, and H the ratio whose hyperbolic logarithm is $\frac{3}{12n} - \frac{15}{360n^3} + \frac{63}{1260n^5}$ * &c.) and the ratio of R$b$$t$Q to RQW$b$ at the point of contrary flexure, will be

$$\frac{n+1}{\sqrt{n}\times\sqrt{n-1}} \times \frac{.797884}{H} \times \overline{1 - \frac{n}{2.3.\overline{n-1}} + \frac{n \times \overline{n-2}}{2.5.4.\overline{n-1}^2} - \frac{n.\overline{n-2}.\overline{n-4}}{2.3\ 7.8.\overline{n-1}^3} + \frac{n.\overline{n-2}.\overline{n-4}.\overline{n-6}}{2.3.4.9.16.\overline{n-1}^4} - \&c.}$$

Now when n is little, the value of this expression will be considerably greater than .6822. It approaches to this continually as n increases; and when n is large, it may be taken for this exactly. Thus when $n = 6$, this expression is equal to .804. When $n = 110$, it is equal to .6903. If we would know the ratio of R$b$$t$Q to RQW$b$, when C$f$ comes to decrease no faster in respect of Qt, than Qt decreases in respect of Ft; that is, when the excess of Qt above Cf, is greatest in comparison of the excess of Ft above Qt, it may be found (by putting the fourth term of the series in the 14th Art. equal to the second term, and then finding the value of z) to be about .8426, when n, p, and q are considerable; and in other cases greater.

Coroll. 'Tis easy to gather from hence that in like manner the greatest part of the area ADH lies between the two ordinates at the points of contrary flexure †.

* Vid. the Second Rule in the Essay, Phil. Transf. Vol. LIII.

† From this Article may be inferred a method of finding at once, without any labour, whereabouts it is reasonable to judge the probability of an unknown event lies, about which a given number of experiments have been made. For when

21. R*ht*Q is greater than the arithmetical mean between D*ht*F and D*hf*C. This appears from the latter part of Art. 19. for what is there proved of the ordinates muſt hold true of the contemporary areas generated by them. And though beyond the points at which the ratio of the decreaſe of Q*t* to the decreaſe of F*t* comes to an equality with the ratio of the decreaſe of Q*t* to the decreaſe of C*f*, the exceſs of F*t* above Q*t* begins to grow larger than before in reſpect of the exceſs of Q*t* above C*f*; yet as it appears from the laſt article, that above five ſixths of the areas RQW*h* and ACFH are generated before the ordinates come to theſe points, and as alſo beyond theſe points the ſaid ratios, 'till they become

neither p nor q are very ſmall, or even not leſs than 10, it will be nearly an equal chance, that the probability of the event lies between $\frac{p}{n}+\frac{\sqrt{pq}}{\sqrt{2n^3-2n^2}}$ and $\frac{p}{n}-\frac{\sqrt{pq}}{\sqrt{2n^3-2n^2}}$. It will be the odds of two to one that it lies between $\frac{p}{n}+\frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$ and $\frac{p}{n}-\frac{\sqrt{pq}}{\sqrt{n^3-n^2}}$; and the odds of five to one that it lies between $\frac{p}{n}+\frac{\sqrt{2pq}}{\sqrt{n^3-n^2}}$ and $\frac{p}{n}-\frac{\sqrt{2pq}}{\sqrt{n^3-n^2}}$. For inſtance; when $p=1000$, $q=100$, there will be nearly an equal chance, that the probability of the event lies between $\frac{10}{11}+\frac{1}{163}$ and $\frac{10}{11}-\frac{1}{163}$; two to one that it lies between $\frac{10}{11}+\frac{1}{115}$ and $\frac{10}{11}-\frac{1}{115}$; and five to one that it lies between $\frac{10}{11}+\frac{1}{81}$ and $\frac{10}{11}-\frac{1}{81}$.

negative and for some time afterwards, are but small; the effect produced before towards rendering the excess of D*ht*F above R*bt*Q always less than the excess of R*bt*Q above D*hf*C, will be such as cannot be compensated for afterwards.

A further proof of this will appear from considering that even when R*bt*Q is increased to RQW*h*, it is but little short of the arithmetical mean between AD*h* and HD*h*. For from Art. 11. and 20. it may be inferred that the ratio of the whole area RQW*h* to this mean, or to $\frac{ACFH}{2}$, is $b \times H$, which is never far from the ratio of equality, but when both p and q are of any considerable magnitude, it is very nearly the ratio of equality. For example; when $n = 110$, $q = 100$, $p = 10$, it is .9938.

22. The ratio of D*ht*F to R*ht*Q is less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to one. For by Art. 12. the ratio of D*ht*F to D*hf*C is less than that of $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ to $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. But by the last Art. R*ht*Q is greater than the arithmetical mean between D*ht*F and D*hf*C, and 1 is exactly the arithmetical mean between $1 + 2Ea^pb^q + \frac{2Ea^pb^q}{n}$ and $1 - 2Ea^pb^q - \frac{2Ea^pb^q}{n}$. From whence this Article is plain.

23. The

23. The ratio of $DhtF$ to $ACFH$ is greater than Σ, and leſs than $\Sigma \times \overline{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}$. For $DhtF$ being greater than $RhtQ$, the ratio of it to $ACFH$ muſt be greater than the ratio of $RhtQ$ to $ACFH$, or greater than Σ. Alſo; ſince the ratio of $RhtQ$ to $ACFH$ is equal to Σ; and the ratio of $DhtF$ to $RhtQ$ is leſs than the ratio of $1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}$ to 1; it follows that the ratio compounded of the ratio of $RhtQ$ to $ACFH$, and of $DhtF$ to $RhtQ$, that is, the ratio of $DhtF$ to $ACFH$ muſt be leſs than $\Sigma \times \overline{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}$.

24. The ratio of $DhtF + DhfC$ to $ACFH$ (that is, the chance for being right in judging that the probability of an event perfectly unknown, which has happened p and failed q times in $p+q$ or n trials, lies ſomewhere between $\frac{p}{n} + z$ and $\frac{p}{n} - z$) is greater than $\frac{2\Sigma}{1 + 2Ea^p b^q + \frac{2Ea^p b^q}{n}}$, and leſs than 2Σ. The former part of this Art. has been already proved, Art. 12. The latter part is evident from Art. 21. For $RhtQ$ being greater than the arithmetical mean between $DhtF$ and $DhfC$, $2RhtQ$ muſt be greater than $DhtF + DhfC$; and conſequently the

the ratio of 2 R*ht*Q to A C F H, greater than the ratio of D*ht*F + D*hf*C to A C F H*.

It will be eaſily ſeen that this Article improves conſiderably the rule given in Art. 12. But we may determine within ſtill narrower limits whereabouts the required chance muſt lie, as will appear from the following Articles.

25. If c and d ſtand for any two fractions, whenever the fluxion of $c \times Ft$ is greater than the fluxion of $d \times Cf$ (Vid. fig.) $c \times Ft + d \times Cf$ will be greater than Qt. For in the ſame manner with Art. 6. it will appear that $c \times Ft + d \times Cf$ is to Qt, as the fluxion of $c \times Ft \times \overline{1+\frac{nz}{q}} \times \overline{1-\frac{nz}{q}}$ together with the fluxion of $d \times Cf \times \overline{1-\frac{nz}{p}} \times \overline{1+\frac{nz}{q}}$ to the fluxion of $Qt \times \overline{1-\frac{n^2z^2}{pq}}$. Now ſince $1-\frac{n^2z^2}{pq}$ is the arithmetical mean between $\overline{1+\frac{nz}{p}} \times \overline{1-\frac{nz}{q}}$ and $\overline{1-\frac{nz}{p}} \times \overline{1+\frac{nz}{q}}$, it is plain, that were the fluxion of $c \times Ft$ equal to the fluxion of $d \times Cf$, $c \times Ft + d \times Cf$ would decreaſe in reſpect of its own magnitude at the ſame rate with Qt; and, therefore, ſince at firſt equal, they would always continue equal. But the fluxion of $c \times Ft$ being greater than the fluxion of $d \times Cf$ by ſuppoſition, and (ſince q greater than) p $\overline{1+\frac{nz}{p}} \times \overline{1-\frac{nz}{q}}$,

* This Art. is true, whether p be greater or leſs than q.

also greater than $\overline{1 - \frac{nz}{p}} \times \overline{1 + \frac{nz}{q}}$, it follows that the fluxion of $c \times Ft \times \overline{1 + \frac{nz}{p}} \times \overline{1 - \frac{nz}{q}}$ added to the fluxion of $d \times Cf \times \overline{1 - \frac{nz}{p}} \times \overline{1 + \frac{nz}{q}}$ is greater than these two fluxions multiplied by $\overline{1 - \frac{n^2 z^2}{pq}}$; and, therefore, greater, than the fluxion of $Qt \times \overline{1 - \frac{n^2 z^2}{pq}}$; and, therefore, $c \times Ft + d \times Cf$ greater than Qt.

26. If we suppose three continued arithmetical means betwen Cf and Ft $\left(\frac{3Cf + Ft}{4}, \frac{Cf + Ft}{2}, \frac{3Ft + Cf}{4}\right)$ Qt will be greater than the second, and less than the third, if p is greater than 1. That Qt will be greater than the second has been already proved; and that it will be less than the third, will be an immediate consequence from the last Article, if it can be shewn that the fluxion of $\frac{3Ft}{4}$ is greater than the fluxion of $\frac{Cf}{4}$. This will appear in the following manner. The ratio of the fluxion of Cf to the fluxion of Ft is by Art. 7. and 14. $\frac{\overline{1 - \frac{nz}{p}}\Big|^{p-1} \times \overline{1 + \frac{nz}{q}}\Big|^{q-1}}{\overline{1 + \frac{nz}{p}}\Big|^{p-1} \times \overline{1 - \frac{nz}{q}}\Big|^{q-1}}$. The hyperbolic logarithm of

of this ratio is $\overline{\frac{1}{p} - \frac{1}{q}} \times 2nz - \overline{\frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{q^2} + \frac{1}{q^3}} \times \frac{2n^3z^3}{3} - \overline{\frac{1}{p^4} - \frac{1}{p^5} - \frac{1}{q^4} + \frac{1}{q^5}} \times \frac{2n^5z^5}{5}$, &c. This ratio by Art. 18. is greateſt at the point of contrary flexure, or when $z = \frac{\sqrt{pq}}{\sqrt{n^3 - n^2}}$. Subſtitute this for z in the ſeries, and it will become $\overline{\frac{1}{p} - \frac{1}{q}} \times \frac{2\sqrt{pq}}{\sqrt{n-1}} - \overline{\frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{q^2} + \frac{1}{q^3}} \times \frac{2p^{\frac{3}{2}} \times q^{\frac{3}{2}}}{3 \times \overline{n-1}|^{\frac{3}{2}}} -$, &c. which, therefore, expreſſes the logarithm of the ratio when greateſt, and will eaſily diſcover it in every caſe. 'Tis apparent that the value of this ſeries is greateſt when p is leaſt in reſpect of q. Suppoſe then $p = 2$, and q infinite. In this caſe, the value of the ſeries will be 1.072, and the number anſwering to this logarithm is not greater than 2.92. The fluxion, therefore, of Cf, when greateſt, cannot be three times the contemporary fluxion of Ft; from whence it follows that the fluxion of $\frac{3Ft}{4}$ muſt be greater than the fluxion of $\frac{Cf}{4}$.

It is eaſy to ſee how theſe demonſtrations are to be varied when q is leſs than p, and how in this caſe ſimilar concluſions may be drawn. Or, the ſame concluſions will in this caſe immediately appear, by changing p into q and q into p, which will not make any difference in the demonſtrations.

In the manner ſpecified in this Article we may always find within certain limits how near the value of Qt comes to the arithmetical mean between Ft and Cf, which limits grow narrower and narrower, as

p and q are taken larger, or their ratio comes nearer to that of equality, 'till at laſt, when p and q are either very great or equal, Qt coincides with this mean. Thus, if either p or q is not leſs than 10; that is, in all caſes, where it is not practicable without great difficulty to find the required chance exactly by the firſt rule, Qt will be greater than the fourth, and leſs than the fifth of ſeven arithmetical means between Cf and Ft.

27. The arithmetical means mentioned in the laſt Article may be conceived as ordinates deſcribing areas at the ſame time with Qt; and what has been proved concerning them is true alſo of the areas deſcribed by them compared with RbtQ.

28. If either p or q is greater than 1, the true chance that the probability of an unknown event which has happened p times and failed q in $\overline{p+q}$ or n trials, ſhould lie ſomewhere between $\frac{p}{n} + z$ and $\frac{p}{n} - z$ is leſs than $2\,\Sigma$, and greaterer than $\Sigma +$

$$\frac{\Sigma \times \overline{1 - 2\,E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}}}{1 + E\,a^p b^q + \frac{E\,a^p b^q}{n}}.$$

If either p or q is greater than 10, this chance is leſs than $2\,\Sigma$, and greater than

$$\Sigma + \frac{\Sigma \times \overline{1 - 2\,E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}}}{1 + \frac{1}{2}\,E\,a^p b^q + \frac{E\,a^p b^q}{2n}}.$$

 This

This is eaſily proved in the ſame manner with Art. 12, 23, 24.

That it may appear how far what has been now demonſtrated improves the ſolution of the preſent problem, let us take the fifth caſe mentioned in the Appendix to the Eſſay, and enquire what reaſon there is for judging that the probability of an event concerning which nothing is known, but that it has happened 100 times and failed 1000 times in 1100 trials, lies between $\frac{10}{11} + \frac{1}{110}$ and $\frac{10}{11} - \frac{1}{110}$. The ſecond rule as given in Art. 12. informs us, that the chance* for this muſt lie between .6512, (or the odds of 186 to 100) and .7700, (or the odds of 334 to 100). But from the laſt Art. it will appear that the required chance in this caſe muſt lie between 2 Σ, and

$$\Sigma + \Sigma \times \frac{1 - E\,a^p b^q - \frac{2\,E\,a^p b^q}{n}}{1 + \frac{1}{10} E a^p b^q + \frac{E\,a^p b^q}{10\,n}};$$ or, between

.6748 and .7057; that is, between the odds of 239 to 100, and 207 to 100.

In all caſes when z is ſmall, and alſo whenever the diſparity between p and q is not great 2 Σ is almoſt exactly the true chance required. And I have reaſon to think, that even in all other caſes, 2 Σ gives the

* In the Appendix, this chance, as diſcovered by Mr. Bayes's ſecond rule, is given wrong, in conſequence of making m^2 equal to $\frac{n^3}{p\,q}$, whereas it ſhould have been taken equal to $\frac{n^3}{2pq}$ as appears from Article 8.

true

true chance nearer than within the limits now determined. But not to purſue this ſubject any further; I ſhall only add that the value of 2 Σ may be always calculated very nearly, and without great difficulty; for the approximations to the value of E $a^p b^q$, and of the ſeries $m - z \frac{m^3 z^3}{3} + \frac{n-2}{2.n} \times \frac{m^5 z^5}{5}$, &c. * given in the Eſſay, are ſufficiently accurate in all caſes where it is neceſſary to uſe them.

* In the expreſſion for this laſt approximation there is an error of the preſs which ſhould be corrected; for the ſign before the fourth term ſhould be — and not +.

LIII. *An*